HL-003-016302 Seat No. ________

M. Sc. (Sem. III) (CBCS) Examination

May / June – 2017

Mathematics - CMT-3002

(Functional Analysis)

(Old Course)

Faculty Code : 003
Subject Code : 016302

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer all questions.
(2) Each question carries 14 marks.
(3) The figures on the right indicate marks allotted to the question.

1 Answer any seven questions : **2×7=14**

(1) If $\alpha, \beta \geq 0$ and p, q are conjugate exponents then prove that $\alpha\,\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$.

(2) Is $\left(C_{00}, \|\cdot\|_\infty\right)$ a Banach space ? Justify.

(3) Prove that $\left(\ell^\infty, \|\cdot\|_\infty\right)$ is not separable.

(4) Does the maximum norm on $C[a, b]$ satisfy parallelogram law ? Justify.

(5) If X is an inner product space over $I\!K$ and $\phi \neq M \subset X$ then prove that $M \subset M^{\perp\perp}$.

(6) If $(x, <, >)$ is an inner product space over $I\!K$ and $u, v \in X$, prove that $\langle x, u\rangle = \langle x, v\rangle, \forall\, x \in X \Rightarrow u = v$.

(7) In a n.l. space over $I\!K$, prove that strong convergence $\Rightarrow$ weak convergence.

(8) If $\|\bullet\|$ is the norm induced by an inner product $\langle , \rangle$ on an inner product space X then prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2, \forall\, x, y \in X, x \perp y$

(9) Define the canonical mapping $C : x \to x''$ and prove that it is an isometry.

(10) Prove that $\|x\|_1 \le \sqrt{n}\, \|x\|_2, \forall x \in I\!K^n$.

2 Answer any two questions : **2×7=14**

(a) Define Banach space over $I\!K$ and prove that $\left(C[0,1], \|\bullet\|_2\right)$ is not a Banach space over $I\!K$.

(b) A n.l. space over $I\!K$ is finite-dimensional iff $\bar{B}(0,1)$ is compact.

(c) Define equivalent norms on a vector space over $I\!K$ and prove that any norms on a finite dimensional vector space X over $I\!K$ are equivalent.

3 (a) Define bdd linear transformation between two n.l. spaces X, Y over $I\!K$. Prove that a linear transformation $J = X \to Y$ is bdd iff it is continuous at $x_0 \in X$. **7**

(b) If X is a n.l. space over $I\!K$ and Y is a Banach space over $I\!K$ then prove that $\left(B(X,Y), \|\bullet\|\right)$ is a Banach space over $I\!K$ where $\|T\| = \inf\left\{c > 0 \,|\, \|Tx\| \le C\|x\|, \forall\, x \in X\right\}$, $\forall\, T \in B(x,y)$ (Prove only completeness) **7**

OR

3 (c) Prove that $\left(\ell^1, \|\bullet\|_1\right)' \cong \left(\ell^\infty, \|\bullet\|_\infty\right)$. **7**

(d) Define Schauder basis in a n.l. space over $I\!K$. Prove that every n.l. space over $I\!K$ with a Schauder basis is separable. **7**

4 Answer any **two** questions : **2×7=14**

(a) Define inner product space, Hilbert space over $I\!K$ and give an example of an inner product space which is not a Hilbert space with justification.

(b) Let X be an inner product space over $I\!K$ and M be a non-empty complete convex subset of X. Then prove that given $x \in X$, $\exists$ a unique $y_0 \in M$ s.t. $d(x, M) = \|x - y_0\|$.

(c) State and prove Bessel's inequality for an orthonormal sequence in an inner product space.

5 Answer any two questions : **2×7=14**

(a) State, without proof, Hahn-Benach theorem for a n.l. space X over $I\!K$. Prove that X' separates points of X.

(b) State, without proof, uniform bddness theorem. Give an example to show that the hypothesis "X is a Banach space" in the uniform bddness theorem can not be dropped.

(c) Define open mapping between two topological spaces. State and prove open mapping theorem.

(d) Prove that the dual of a Hilbert space is a Hilbert space.